

Math 150: Functions

Jorge Alberto Calvo

Department of Mathematics and Physics
Ave Maria University
Ave Maria, Florida, USA

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Prayer of St. Thomas Aquinas

Oh merciful God, grant that I may
ardently desire,
prudently examine,
truthfully acknowledge,
and perfectly accomplish
what is pleasing to You,
for the praise and glory of Your Name.

Amen.

To Angelo DiDomenico and Ken Millett
for their lessons, advice, and friendship.

Apology to the Student

Welcome to *Mathematics 150* at Ave Maria University! This course is centered around the concept of **function**, a mathematical idea that has become increasingly important over the past two centuries. Throughout this semester, you will learn about many of the topics commonly covered in courses with impressive-sounding titles like *College Algebra*, *Trigonometry*, and *Precalculus*. A well-prepared student will already be familiar with the basic concepts of arithmetic, geometry, and algebra. In particular, it will be expected that the reader already knows how (and how not) to manipulate fractions, multiply positive and negative numbers, and solve very simple equations. It is our hope that students completing this course will be prepared (and perhaps even want) to move on to a course in Calculus. However, you should be warned that for a variety of reasons this course is probably different than any course in mathematics that you may have taken in the past, either in high school or even at another college or university.

First of all, Math 150 is a **university-level course**; this means that we have higher expectations of you, both in the amount of material that you need to learn and in the level at which you need to understand it. At times, you may feel overwhelmed by the fast pace of the course or the large number of homework exercises you will be asked to complete each night. To help you succeed, you will have access to tutors and to your instructor, both in class and in office hours. However, the responsibility to keep up with the reading and the homework and to attend class on a regular basis falls squarely on *your* shoulders. Do not make the mistake of thinking that you can learn everything you need to know on the night before an exam.

Secondly, Math 150 has been designed as a part of the **core curriculum** emphasizing the liberal arts identity of Ave Maria University. This means that these course notes were written with an eye to highlight mathematics as an *art form*. Although it may come as a surprise to those students who were beaten down as youths with such mental tortures as multiplication tables and long division, mathematics is chock-full of beautiful ideas reflecting a perfect balance between creativity and logical consistency. These notes will take you on a journey from the ancient problem of computing the square root of two to the complex affair of visualizing an imaginary number like the square root of negative one. Along the way, you will come across several historical figures, each of whom made his own contribution to the human endeavor that is the story of mathematics. The point of all of this is that you *will* be expected to read these notes, and to read them carefully.

As you quickly skim through the first few pages of these notes, you will notice that they are more than just a collection of examples, each working out a different type of problem, which you are to mimic in the homework exercises. The examples have been carefully chosen to give you an incrementally clearer picture of the concepts at hand and of how you should be thinking about them. By the same token, some of the exercises in the problem sets will not look like any of the examples. Instead, some will ask you to perform a mathematical experiment; others will ask you to use your imagination and creativity to propose a reasonable conjecture. All of them will ask you to think. Be sure to give yourself plenty of time to work on these exercises.

You should read these notes slowly, a few sentences at a time. Take your time. Make sure you understand what you read. If you don't, go back a few sentences and try again. Take an active role in the reading by working out the examples and by asking yourself why one statement connects with the next. Remember that by approaching the reading as a mystery to be unraveled, rather than just as information and procedures to be memorized, you will bring out the best of your intellectual abilities until, ultimately, the mathematics itself will become clearer, more meaningful, and more beautiful.

As a final note, I would like to stress that these notes are a work in progress. I have tried to stamp out all typographical errors, but one or two may have stubbornly survived all of my proofreading. If you find an error, please let me know! Furthermore, a lot of effort has been put into making these notes interesting and, yes, sometimes even entertaining reading. Whether I succeeded or not is a matter of opinion. Nevertheless, I would consider it quite an insult if you did not at least giggle politely to yourself every once in a while. Now would be a good time to begin.

Good luck. And may all your problems be math problems!

J. A. C.
Naples, FL
August 2008

Table of Contents

Chapter 1. An Introduction To Functions

- 1.1 What Are Functions?
- 1.2 Equations
- 1.3 Inequalities
- 1.4 Functions and Formulas
- 1.5 Cartesian Coordinates
- 1.6 Functions and Graphs

Chapter 2. Polynomials

- 2.1 Linear Functions
- 2.2 Regression
- 2.3 Linear Transformations
- 2.4 Quadratic Functions
- 2.5 Systems of Equations

Chapter 3. Exponentials and Logarithms

- 3.1 Exponential Functions
- 3.2 Rates of Change Revisited
- 3.3 Logarithmic Functions
- 3.4 Logarithms Up Close And Personal
- 3.5 Exponential Growth and Decay

Chapter 4. Trigonometry

- 4.1 Angles
- 4.2 Sine, Cosine, Tangent
- 4.3 Waveforms
- 4.4 Oblique Triangles

Chapter 5. The Complex Numbers

- 5.1 The Square Root of Negative One
- 5.2 The Argand Plane
- 5.3 The Formulas of DeMoivre and Euler
- 5.4 Epilogue

Index

- π , 8, 14, 41
- \emptyset , *see also* empty set
- ϕ , *see also* golden ratio
- ∞ , *see also* infinity

- absolute value function, **8**, 42, 63, 92
- algebra, 16, 26
- algorithm, 39
- Al-Jabr wa'l-Muqabala*, 16
- Al-Khwarizmi, Abu, 16, 39
- Apollonius of Perga, 121
- arithmetic properties
 - associative, 15
 - commutative, 15
 - distributive, 17
- arithmetic sequence, **91**
 - degree two, 120
- average rate of change, **77**

- Babylonian algorithm, 39
- bar graph, 6
- bubble diagram, 3

- Cantor, Georg, 25
- cartesian coordinate system, **47**
- cartesian plane, 47
- causation, **97**
- codomain, **3**, 42
- coefficient, **18**
- completing the square, 54
- composition of functions, 113
- concavity, **121**
- conic sections, 121
- constant function, **35**
- correlation, **97**
- cube root function, 42

- decimal representation, 13
- decreasing graph, 62
- degree
 - of polynomial, **36**
 - of power function, **35**
- demand function, 91, 135
- Descartes, René, 47, 56
- difference of two squares, 18
- difference quotient, **77**

- dilation, **107**
- directrix, 123
- Dirichlet's function, 8, 68
- Dirichlet, Peter Gustav Lejeune, 68
- discriminant, **129**
- distance formula, 53
- division by zero, 13, 42
- domain, **3**, 42

- empty set, 32
- equation, **16**
 - circle, 54
 - line, 48, 86
 - quadratic, 129
- equilibrium price, 135
- evaluation of functions, 5

- factoring
 - algebraic expressions, 18–21
 - square roots, 40
- Fibonacci, Leonardo, 6
- focus, 123
- formula, **6**, 35
 - distance, 53
 - point-slope, 84
 - quadratic, 127
 - recursive, 8
 - regression, 95
 - slope-intercept, 80
- function, **3**
 - composition, 113
 - description
 - approximation, 40
 - bar graph, 6
 - bubble diagram, 3
 - formula, 6, 35
 - graph, 61
 - recursive formula, 8
 - scatter plot, 93
- evaluation, 5
- notation, 5
- types
 - absolute value, 8, 42, 63, 92
 - constant, 35
 - identity, 6

- linear, 79
- piecewise-defined, 8
- polynomial, 36
- power, 35
- quadratic, 120
- radical, 42
- rational, 36
- zero, 35
- well-defined, 8

Gauss, Carl Friedrich, 94, 119

golden ratio, 14, 21, 68, 124, 130

graph

- of equation, **48**
- of function, **61**

Harmonices Mundi, 102

Heron of Alexandria, 39

Hippasus of Metapontum, 15

horizontal

- dilation, 108
- reflection, 103
- translation, 110

hyperbola, 50, 65

hypotenuse, 51

identity function, **6**

image, *see also* function evaluation

image set, **4**

increasing graph, 61

indices, 17

inequality, **26**

infinity, 25

integer, **13**

intercept, **51**, 67

interval, **25**

inverse relationship, 40

irrational number, 8, **14**, 25, 40

Kepler, Johannes, 102

law of supply and demand, 136

least squares regression, **94**

Legendre, Adrien-Marie, 94

Liber Abaci, 6

line of best fit, *see also* least squares regression

linear correspondence, 138

linear function, **79**

linear transformation, **106**

- dilation, 107
- reflection, 103
- translation, 110

local maximum, 62

local minimum, 62

lowest terms

- fraction, 13
- rational function, 43

map, *see also* function evaluation

Maple, 98

marginal cost, 78

method

- completing the square, 54
- Newton's, 42
- sledgehammer, 19
- square root, 41
- substitution, 136

model functions, 63–65

\mathbb{N} , 13

Newton's method, 42

notation

- function, 5
- interval, 25
- set-builder, 13

n th root function, 42

number

- integer, 13
- irrational, 8, 14, 25, 40
- rational, 8, 13, 25
- real, 6, 13, 25

obelus, 15

order of operations, 15

ordered pair, 47

origin, 47

parabola, 49, 65, **121**

perfect square, 18, 40

piecewise-defined function, **8**

point reflection, 104

point-slope formula, **84**

polynomial function, **36**

power function, **35**

problem solving, 140

- mixture problem, 141
- optimization problem, 146
- rate problem, 142
- revenue problem, 145

proof, *see also* theorem

properties

- associative, 15
- commutative, 15
- distributive, 17

proportionality, 78

Pythagoras of Samos, 51

Pythagoras's Theorem, 52

\mathbb{Q} , 13

quadrant, **48**

quadratic equation, 129

quadratic formula, **127**

quadratic function, **120**

\mathbb{R} , 13

radical function, **42**

range, 4

rate of change

- average, 77

rational function, **36**

rational number, 8, **13**, 25

real line, 25

real number, 6, **13**, 25
 recursive formula, 8
 reflection, **103**
 regression line, *see also* least squares regression
 ruler postulate, 25
 rules
 algebra for equations, 16
 algebra for inequalities, 26

 scatter plot, **93**
 sequence
 arithmetic, 91
 degree two, 120
 sledgehammer method, 19
 slope, **80**, 86
 slope-intercept formula, **80**
 solidus, 15
 solution
 of equations, **16**
 of inequalities, **26**
 of system of equations, 136
 solve, *see also* solution
 square root
 of negative numbers, 39, 42
 of two, 15, 38
 square root function, 39
 square root method, 41
 stabilizing digits, 38
 start-up cost, 79
 substitution method, 136
 supply function, 135
 syntax, 99
 system of equations, **136**

 theorem, 17
 factoring, 18
 graph of linear function, 80
 graph of quadratic function, 124
 least squares regression, 95
 Pythagoras's, 52
 vertical line test, 66
 transformation, *see also* linear transformation
 translation, **110**

 union, 28

 variable, **6**
 vertex, **123**
 vertical
 dilation, 107
 line test, 66
 reflection, 103
 translation, 110

 well-defined, **8**

x-axis, 47
x-coordinate, 47
x-intercept, *see also* intercept
xy-plane, *see also* cartesian plane

y-axis, 47
y-coordinate, 47
y-intercept, *see also* intercept

 \mathbb{Z} , 13
 zero function, **35**